

# Note on Cyclic Sum and Combination Sum of Color-ordered Gluon Amplitudes

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**ABSTRACT:** Continuing our previous study [22] of permutation sum of color ordered tree amplitudes of gluons, in this note, we prove the large- $z$  behavior of their cyclic sum and the combination of cyclic and permutation sums under BCFW deformation. Unlike the permutation sum, the study of cyclic sum and the combination of cyclic and permutation sums is much more difficult. By using the generalized Bern-Carrasco-Johansson (BCJ) relation, we have proved the boundary behavior of cyclic sum with nonadjacent BCFW deformation. The proof of cyclic sum with adjacent BCFW deformation is a little bit simpler, where only Kleiss-Kuijf (KK) relations are needed. Finally we have presented a new observation for partial-ordered permutation sum and applied it to prove the boundary behavior of combination sum with cyclic and permutation.

**KEYWORDS:** Gauge Symmetry , Duality in Gauge Field Theories.

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## 1. Introduction

On-shell recursion relation for tree-level gluon amplitudes [1, 2] has been shown to be important not only for real calculations, but also for theoretical understanding of many important properties such as the BCJ relation [3]<sup>1</sup> and Kawai-Lewellen-Tye relation [8, 9, 10, 11, 12]<sup>2</sup>. To establish the on-shell recursion relation, understanding of large- $z$  behavior (or the “boundary behavior”) of amplitude under BCFW-deformation on a chosen pair of particles  $(i, j)$

$$p_i \rightarrow p_i - zq, \quad p_j \rightarrow p_j + zq, \quad q^2 = q \cdot p_i = q \cdot p_j = 0 \quad (1.1)$$

becomes crucial. However estimating boundary behavior is not so easy and a naive analysis from Feynman diagrams could often lead to wrong conclusions. A careful analysis was done by Arkani-hamed and Kaplan

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<sup>1</sup>The BCJ relation has first been proved in string theory [4, 5], and then in field theory [6, 7].

<sup>2</sup>See also recent review [13].

in [14], where because  $zq \rightarrow \infty$ , the whole amplitude can be considered as scattering of particles  $i, j$  from soft background constructed by other particles. For practical purposes it is considerably simpler if  $A(z) \rightarrow 0$  as  $z \rightarrow \infty$ , so that the boundary contribution is simply zero. Furthermore, if the amplitude has even better asymptotic behavior  $A(z) \sim \frac{1}{z^k}$ ,  $k \geq 2$ , it is possible to derive more relations in addition to the standard on-shell recursion relation. These “bonus” relations were discussed in [15, 14, 16, 17, 6, 18, 19], where their usefulness was demonstrated from various aspects.

Because of its importance, it is desirable to have better understanding to this problem. Recently, the boundary behavior of gluon amplitude under deformation (1.1) has been carefully studied by Boels and Isermann in [20, 21], where some new behaviors were observed. Among them, the following two statements are particularly intriguing:

$$\sum_{\text{perm } \alpha} A_n(i, \{\alpha\}, j, \{\beta\}) \rightarrow \xi_{i\mu}(z) \xi_{j\nu}(z) \frac{G^{\mu\nu}(z)}{z^k}, \quad k = \begin{cases} n_\alpha, & i, j \text{ not nearby} \\ n_\alpha - 1, & i, j \text{ nearby} \end{cases} \quad (1.2)$$

and<sup>3 4</sup>

$$\sum_{\text{cyclic } \alpha} A_n(i, \{\alpha\}, j, \{\beta\}) \rightarrow \xi_{i\mu}(z) \xi_{j\nu}(z) \frac{G^{\mu\nu}(z)}{z^k}, \quad k = \begin{cases} 2, & i, j \text{ not nearby} \\ 1, & i, j \text{ nearby} \end{cases} \quad (1.3)$$

where  $n_\alpha$  is the number of elements in set  $\alpha$ ,  $\xi$  is the polarization vector and  $G_{\mu\nu}$  is given by [14]

$$G_{\mu\nu} = z\eta^{\mu\nu} f(1/z) + B^{\mu\nu}(1/z) + \mathcal{O}(1/z). \quad (1.4)$$

These two observations can combine together and following sum has the large- $z$  behavior

$$\sum_{\sigma \in Z(\{2, \dots, i-1\})} \sum_{\rho \in P(\{i+1, \dots, n\})} A(\widehat{1}, \sigma, \widehat{i}, \rho) \rightarrow \xi_{1\mu}(z) \xi_{i\nu}(z) \frac{G^{\mu\nu}(z)}{z^{n-i+1}}, \quad (1.5)$$

i.e., it is one  $\frac{1}{z}$  suppressing comparing to the pure permutation sum of (1.2).

The above results by Boels and Isermann were observed from explicit analysis of Feynman diagrams. Despite intuitive and conceptually straightforward, such approach requires order by order cancelations, which is unfortunately difficult to show and the fundamental mechanism behind these cancelations and the better convergent behavior is far from transparent. Furthermore, as we have mentioned, better divergent behavior could often imply extra relations among amplitudes, such as BCJ and KLT relations, it is crucial and natural to ask *do these two new statements lead to some un-discovered nontrivial relations among color-ordered tree amplitudes?*

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<sup>3</sup>The second statement requires the number in the set  $\alpha$  to be equal or larger than two.

<sup>4</sup>As mentioned by Boels and Isermann in [21], though the behavior of permutation sum and combination sum could be checked up to and including  $z^{-2}$ , the general proofs of these behaviors were hard to given in this way. We should also notice that the adjacent cyclic sum is considered as a special case of combination sum in [21]. However, the non-adjacent cyclic sum cannot be regarded as a special case of the combination sum and this case was not checked in [21].

To better understand the reasons behind cancelations found by Feynman diagram analysis and answer the question raised in previous paragraph, we study boundary behavior of color-ordered amplitudes from another point of view. In [22] we have studied the first statement (1.2) and showed that by using Kleiss-Kuijf (KK) relation and fundamental BCJ relation, statement (1.2) can be derived easily. In this note we continue our study on the second statements (1.3), (1.5). Surprisingly we find that comparing to the first statement, the second statement is technically much more difficult to investigate. Besides the familiar Kleiss-Kuijf (KK) relation and fundamental BCJ relation, an extensive application of generalized BCJ relation[7] is required<sup>5</sup>. This is because fewer symmetries are possessed by the cyclic sum as opposed to those by the permutation sum, and therefore the greater technical challenge is involved. To be able to understand these new nontrivial technical points, before each general proof we provide an example to demonstrate in details.

Our results make the following statements more transparent: First, the observed cancelations in Feynman diagrams after the cyclic or permutation sum are natural consequences of the well known boundary behavior under the BCFW-deformation. Secondly, since the bonus relations of a pair deformation, i.e., the BCJ relations and their generalizations, have been found, there are no new nontrivial bonus relations implied by the better boundary behavior of cyclic or permutation sum.

We would like to emphasize that although there are no new bonus relations for cyclic or permutation sum, *there do exist many nontrivial applications for cyclic or permutation sum*. Some nontrivial examples have been given in [20, 21], where vanishing of box, triangle or bubble coefficients of one-loop amplitudes has been understood from this point of view. Thus it is desirable to study boundary behavior for other cyclic and permutation combinations. A new result by our method is that we have observed the large- $z$  behavior of another type of sum, i.e., the partial-ordered permutation sum

$$\sum_{\sigma \in P(\{2, \dots, l\} \cup \{l+1, \dots, i-1\})} A(\hat{1}, \sigma, \hat{i}, \dots, n) \sim \frac{1}{z} \sum_{\sigma' \in P(\{l+1, \dots, i-1\})} A(\hat{1}, \sigma', \hat{i}, \dots, n) \quad (1.6)$$

where the sum is over all permutations of elements  $\{2, 3, \dots, i-1\}$  under one condition: the relative ordering in the subset  $\{2, \dots, l\}$  is kept. An application of our new result (1.6) is to prove (1.5) given by Boels and Isermann. Other possible applications of (1.6) could be on simplifying coefficients of loop amplitudes. Also since our proof uses only KK and (generalized) BCJ relations, which are also true for  $\mathcal{N} = 4$  SYM theory, all results in this note are automatically true for  $\mathcal{N} = 4$  SYM theory. Finally, it is natural to generalize our method to other situations, such as to string theory [26] or Witten's diagram [27] where BCFW on-shell recursion relation has been applied.

The plan of this note is the following: In section two, we prove the large- $z$  behavior for cyclic sum under non-adjacent BCFW deformation while in section three we prove the large- $z$  behavior for cyclic sum under adjacent BCFW deformation. We present the proof for a new observation concerning the partial-ordered

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<sup>5</sup>KK relation and generalized BCJ relation in fact can be generated by only using fundamental BCJ relation in addition with cyclic symmetry[23].

permutation sum (1.6) in section four and finally using the result in section five we prove the combination sum given by (1.5).

### 1.1 Some backgrounds

For self-completeness we review the formulas needed in our proofs. The first one is the Kleiss-Kuijf (KK) relation, which was first conjectured in [24] and later proved in [25]. The formula reads

$$A_n(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in P(O\{\alpha\} \cup O\{\beta^T\})} A_n(1, \sigma, n), \quad (1.7)$$

where the  $P(O\{\alpha\} \cup O\{\beta^T\})$  sum is to be taken over all permutations of set  $\alpha \cup \beta^T$  whereas the relative ordering in sets  $\alpha$  and  $\beta^T$  (which is the reversed orderings of set  $\beta$ ) are preserved. The  $n_\beta$  here is the number of elements in set  $\beta$ . One non-trivial example with six gluons is given as the following

$$\begin{aligned} A(1, \{2, 3\}, 6, \{4, 5\}) &= A(1, 2, 3, 5, 4, 6) + A(1, 2, 5, 3, 4, 6) + A(1, 2, 5, 4, 3, 6) \\ &+ A(1, 5, 4, 2, 3, 6) + A(1, 5, 2, 4, 3, 6) + A(1, 5, 2, 3, 4, 6). \end{aligned} \quad (1.8)$$

The second formula we need is the generalized BCJ relation given by [7]

$$\sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \sum_{i=1}^{n_\beta} \sum_{\xi_{\sigma(J)} < \xi_{\sigma(\beta_i)}} s_{\beta_i J} A_n(1, \{\sigma\}, n) = 0, \quad (1.9)$$

where  $(n-2)$ 's elements have been divided into two subsets  $\alpha, \beta$  arbitrarily. In the sum, the position of an element  $t$  in a given ordering  $\sigma$  is denoted by  $\xi_{\sigma(t)}$  with the convention that the position of particle 1 is defined as  $\xi_{\sigma(1)} = 0$ , thus the sum  $\sum_{\xi_{\sigma(J)} < \xi_{\sigma(\beta_i)}} s_{\beta_i J}$  is over all elements at the left hand side of  $i$ -th element in the set  $\beta$ . One example with six gluons is given as following

$$\begin{aligned} 0 &= [(s_{41} + s_{42} + s_{43}) + (s_{51} + s_{52} + s_{53} + s_{54})] A(1, 2, 3, 4, 5, 6) \\ &+ [(s_{41} + s_{42}) + (s_{51} + s_{52} + s_{53} + s_{54})] A(1, 2, 4, 3, 5, 6) \\ &+ [(s_{41} + s_{42}) + (s_{51} + s_{52} + s_{54})] A(1, 2, 4, 5, 3, 6) \\ &+ [(s_{41}) + (s_{51} + s_{52} + s_{53} + s_{54})] A(1, 4, 2, 3, 5, 6) \\ &+ [(s_{41}) + (s_{51} + s_{52} + s_{54})] A(1, 4, 2, 5, 3, 6) + [(s_{41}) + (s_{51} + s_{54})] A(1, 4, 5, 2, 3, 6) \end{aligned} \quad (1.10)$$

where the set  $\alpha = \{2, 3\}$  and the set  $\beta = \{4, 5\}$ . The generalized BCJ relations will be used extensively in our current paper.

## 2. The non-adjacent case of cyclic sum

In this section we prove the large- $z$  behavior of cyclic sum for non-adjacent case, i.e.,

$$\sum_{\text{cyclic } (2, \dots, i-1)} A_n(\widehat{1}, \{2, \dots, i-1\}, \widehat{i}, i+1, \dots, n) \rightarrow \xi_{1\mu}(z) \xi_{i\nu}(z) \frac{G^{\mu\nu}(z)}{z^2} \quad (2.1)$$

with  $4 \leq i \leq n-1$ . For the case  $i = 4$ , the cyclic sum of two elements is same as the permutation sum of them and the large- $z$  behavior can be read off from (1.2) directly, which is exactly (2.1) and has been proved in [22]. Having established that (2.1) is true for  $i = 4$ , we will try to prove general  $i$  by induction.

Comparing to the proof given in [22], the behavior of cyclic sum is not as good as the behavior of permutation sum. Thus the method in [22] can not be applied directly. New inputs must be cooperated, such as generalized BCJ relations and how to split the cyclic sum into different types according to their boundary behaviors. Because these complexities, before giving the general proof, we will use the example  $i = 5$  to demonstrate our idea.

## 2.1 The example with $i = 5$

The first step is to use the generalized BCJ relation to rewrite each term in the summation (2.1). For example, for  $A(\widehat{1}, 2, 3, 4, \widehat{5}, 6, \dots, n)$ , if we choose the set  $\beta = \{2, 3, 4\}$  and the set  $\alpha = \{5, 6, \dots, n-1\}$ , then the generalized BCJ relation (1.9) can be written as following

$$\begin{aligned}
0 = & s_{\widehat{1}234} A(\widehat{1}, 2, 3, 4, \widehat{5}, 6, \dots, n-1, n) + \sum_{k=5}^{n-1} (s_{\widehat{1}23} + \sum_{i=1}^k s_{4i}) A(\widehat{1}, 2, 3, \widehat{5}, \dots, k, 4, k+1, \dots, n) \\
& + \sum_{5 \leq k_1 \leq k_2 \leq n-1} (s_{\widehat{1}2} + \sum_{i=1, i \neq 4}^{k_1} s_{3i} + \sum_{i=1}^{k_2} s_{4i}) A(\widehat{1}, 2, \widehat{5}, \dots, k_1, 3, \dots, k_2, 4, \dots, n) \\
& + \sum_{5 \leq k_1 \leq k_2 \leq k_3 \leq n-1} ( \sum_{i=1, i \neq 3, 4}^{k_1} s_{2i} + \sum_{i=1, i \neq 4}^{k_2} s_{3i} + \sum_{i=1}^{k_3} s_{4i} ) A(\widehat{1}, \widehat{5}, \dots, k_1, 2, \dots, k_2, 3, \dots, k_3, 4, \dots, n) \quad (2.2)
\end{aligned}$$

where it is important to notice that the sum like  $\sum_{i=1}^k s_{4i}$  is independent of  $z$ . Using (2.2) we can solve

$$\begin{aligned}
A(\widehat{1}, 2, 3, 4, \widehat{5}, 6, \dots, n-1, n) &= T_1 + T_2 + T_3 \\
T_1 &= \frac{s_{\widehat{1}23}}{-s_{\widehat{1}234}} \sum_{k=5}^{n-1} A(\widehat{1}, 2, 3, \widehat{5}, \dots, k, 4, k+1, \dots, n) + \frac{s_{\widehat{1}2}}{-s_{\widehat{1}234}} \sum_{5 \leq k_1 \leq k_2 \leq n-1} A(\widehat{1}, 2, \widehat{5}, \dots, k_1, 3, \dots, k_2, 4, \dots, n) \\
T_2 &= \frac{1}{-s_{\widehat{1}234}} \left\{ \sum_{5 \leq k_1 \leq k_2 \leq k_3 \leq n-1} ( \sum_{i=1, i \neq 3, 4}^{k_1} s_{2i} + \sum_{i=1, i \neq 4}^{k_2} s_{3i} + \sum_{i=1}^{k_3} s_{4i} ) A(\widehat{1}, \widehat{5}, \dots, k_1, 2, \dots, k_2, 3, \dots, k_3, 4, \dots, n) \right\} \\
T_3 &= \frac{1}{-s_{\widehat{1}234}} \left\{ \sum_{k=5}^{n-1} ( \sum_{i=1}^k s_{4i} ) A(\widehat{1}, 2, 3, \widehat{5}, \dots, k, 4, k+1, \dots, n) \right. \\
& \left. + \sum_{5 \leq k_1 \leq k_2 \leq n-1} ( \sum_{i=1, i \neq 4}^{k_1} s_{3i} + \sum_{i=1}^{k_2} s_{4i} ) A(\widehat{1}, 2, \widehat{5}, \dots, k_1, 3, \dots, k_2, 4, \dots, n) \right\}, \quad (2.3)
\end{aligned}$$

where we have split all contributions into three types according to their large- $z$  behavior. For the  $T_3$  part, since 1, 5 are not nearby and there is factor  $\frac{1}{-s_{\widehat{1}234}}$ , the large- $z$  behavior will be  $\xi_{1\mu}(z)\xi_{5\nu}(z)\frac{G^{\mu\nu}(z)}{z^2}$ , which

is the prediction given in (2.1), thus it is safe to neglect this part. The naive large- $z$  behavior for  $T_1, T_2$  parts is  $\xi_{1\mu}(z)\xi_{5\nu}(z)\frac{G^{\mu\nu}(z)}{z^2}$  and we need to investigate further.

At the next step we show that after iterations  $T_1$  can be reduced to the sum of forms  $\frac{a}{-s_{\widehat{1}234}}A(\widehat{1}, \dots, \widehat{5}, \dots)$  and forms  $\frac{b}{-s_{\widehat{1}234}}A(\widehat{1}, \widehat{5}, \dots)$  where  $a, b$  are both independent of  $z$ . To see it, we will use similar generalized BCJ relation like the one given in (2.2) for  $A(\widehat{1}, 2, 3, \widehat{5}, \dots, k, 4, k+1, \dots, n)$  with the set  $\beta = \{2, 3\}$  and  $A(\widehat{1}, 2, \widehat{5}, \dots, k_1, 3, \dots, k_2, 4, \dots, n)$  with the set  $\beta = \{2\}$ , thus we can solve

$$A(\widehat{1}, 2, \widehat{5}, \dots, k_1, 3, \dots, k_2, 4, \dots, n) = \frac{1}{-s_{\widehat{1}2}} \sum_t b_t A(\widehat{1}, \widehat{5}, \dots, 2, \dots, n) \quad (2.4)$$

and similarly

$$\begin{aligned} A(\widehat{1}, 2, 3, \widehat{5}, \dots, k, 4, k+1, \dots, n) &= \frac{s_{\widehat{1}2}}{-s_{\widehat{1}23}} A(\widehat{1}, 2, \widehat{5}, \dots, 3, \dots) + \sum \frac{b}{-s_{\widehat{1}23}} A(\widehat{1}, \widehat{5}, \dots, 2, \dots, 3, \dots) \\ &\quad + \sum \frac{a}{-s_{\widehat{1}23}} A(\widehat{1}, 2, \widehat{5}, \dots, 3, \dots, n) \\ &= \sum \frac{b}{-s_{\widehat{1}23}} A(\widehat{1}, \widehat{5}, \dots, 2, \dots, 3, \dots) + \sum \frac{a}{-s_{\widehat{1}23}} A(\widehat{1}, 2, \widehat{5}, \dots, 3, \dots, n) \end{aligned} \quad (2.5)$$

where the first term at the right hand side of first line in (2.5) has been reduced further using (2.4). Putting (2.5) and (2.4) back to  $T_1$ , we see that  $T_1$  reduces to the form we claimed.

Having understood  $T_1$ , Equation (2.3) can be written as the following sum

$$A(\widehat{1}, 2, 3, 4, \widehat{5}, 6, \dots, n-1, n) = \sum_t \frac{a_t}{-s_{\widehat{1}234}} A(\widehat{1}, \dots, \widehat{5}, \dots) + \sum_t \frac{b_t}{-s_{\widehat{1}234}} A(\widehat{1}, \widehat{5}, \dots) \quad (2.6)$$

where the first part has the right large- $z$  behavior and can be neglected. To show the conjecture (2.1), our remaining task is to show that the cyclic sum of the second part of (2.6) is zero.

Now we work out the  $\frac{b_t}{-s_{\widehat{1}234}}A(\widehat{1}, \widehat{5}, \dots)$  part for  $A(\widehat{1}, \sigma_2, \sigma_3, \sigma_4, \widehat{5}, 6, \dots, n)$  where  $\sigma_{2,3,4}$  is reordering of  $(2, 3, 4)$ . First it is easy to see that all  $A$  will be the form that  $\sigma_{2,3,4}$  are inserted between 5 and  $n$  while keeping the ordering of  $(6, \dots, n-1)$ . The relative ordering of  $\sigma_2, \sigma_3, \sigma_4$  can be arbitrary and the  $z$ -independent factor  $b$  is given as following:

- **Type**  $A(\widehat{1}, \widehat{5}, \dots, k_1, \sigma_2, \dots, k_2, \sigma_3, \dots, k_3, \sigma_4, \dots, n)$ : This type can come from several places. The first place is from the  $T_2$  part of (2.3) and the coefficient  $b$  is given by

$$b_I \begin{bmatrix} \sigma_2 & \sigma_3 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = \sum_{i=1, i \neq \sigma_3, \sigma_4}^{k_1} s_{\sigma_2 i} + \sum_{i=1, i \neq \sigma_4}^{k_2} s_{\sigma_3 i} + \sum_{i=1}^{k_3} s_{\sigma_4 i} \quad (2.7)$$

where the first parameter  $(\sigma_2, \sigma_3, \sigma_4)$  gives the relative ordering of these three elements and the second parameter  $(k_1, k_2, k_3)$  tells which elements at the nearest left hand side of corresponding  $\sigma_i$ . The second place is from the  $A(\widehat{1}, \sigma_2, \widehat{5}, \dots, n)$  in  $T_1$  of (2.3) with another solving using the generalized BCJ relation as given in (2.4). To make it clear, we say that the path of second contribution is

$$\text{Path}_{II} \equiv (\widehat{1}, \sigma_2, \sigma_3, \sigma_4, \widehat{5}) \rightarrow (\widehat{1}, \sigma_2, \widehat{5}, \sigma_3, \sigma_4) \rightarrow (\widehat{1}, \widehat{5}, \sigma_3, \sigma_4) \quad (2.8)$$

while the path of the first contribution is

$$\text{Path}_I \equiv (\widehat{1}, \sigma_2, \sigma_3, \sigma_4, \widehat{5}) \rightarrow (\widehat{1}, \widehat{5}, \sigma_2, \sigma_3, \sigma_4) \quad (2.9)$$

The corresponding coefficient of second path is

$$b_{II} \begin{bmatrix} \sigma_2 & \sigma_3 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = - \sum_{i=1, i \neq \sigma_3, \sigma_4}^{k_1} s_{\sigma_2 i} \quad (2.10)$$

The third contribution comes from the path

$$\text{Path}_{III} \equiv (\widehat{1}, \sigma_2, \sigma_3, \sigma_4, \widehat{5}) \rightarrow (\widehat{1}, \sigma_2, \sigma_3, \widehat{5}) \rightarrow (\widehat{1}, \widehat{5}, \sigma_2, \sigma_3) \quad (2.11)$$

and is given by

$$b_{III} \begin{bmatrix} \sigma_2 & \sigma_3 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = - \sum_{i=1, i \neq \sigma_3, \sigma_4}^{k_1} s_{\sigma_2 i} - \sum_{i=1, i \neq \sigma_4}^{k_2} s_{\sigma_3 i} \quad (2.12)$$

The fourth contribution comes from the path

$$\text{Path}_{IV} \equiv (\widehat{1}, \sigma_2, \sigma_3, \sigma_4, \widehat{5}) \rightarrow (\widehat{1}, \sigma_2, \sigma_3, \widehat{5}) \rightarrow (\widehat{1}, \sigma_2, \widehat{5}) \rightarrow (\widehat{1}, \widehat{5}) \quad (2.13)$$

and is given by

$$b_{IV} \begin{bmatrix} \sigma_2 & \sigma_3 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = + \sum_{i=1, i \neq \sigma_3, \sigma_4}^{k_1} s_{\sigma_2 i} \quad (2.14)$$

Summing these four contribution together we have

$$b^{(\sigma_2, \sigma_3, \sigma_4)} \begin{bmatrix} \sigma_2 & \sigma_3 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = \sum_{i=1}^{k_3} s_{\sigma_4 i} \quad (2.15)$$

where the superscript tells that the original amplitude is the ordering  $A(\widehat{1}, \sigma_2, \sigma_3, \sigma_4, \widehat{5}, \dots, n)$ .

- **Type  $A(\widehat{1}, \widehat{5}, \dots, k_1, \sigma_3, \dots, k_2, \sigma_2, \dots, k_3, \sigma_4, \dots, n)$ :** There are several pathes giving contributions. The first one is from path  $\text{Path}_{II}$  as

$$b_{II} \begin{bmatrix} \sigma_3 & \sigma_2 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = - \sum_{i=1, i \neq \sigma_3}^{k_2} s_{\sigma_2 i} \quad (2.16)$$

The second contribution is from the fourth path as

$$b_{IV} \begin{bmatrix} \sigma_3 & \sigma_2 & \sigma_4 \\ k_1 & k_2 & k_3 \end{bmatrix} = \sum_{i=1, i \neq \sigma_3}^{k_2} s_{\sigma_2 i} \quad (2.17)$$

Adding them up we get zero for this type.



- **Type  $A(\widehat{1}, \widehat{5}, \dots, k_1, \sigma_3, \dots, k_2, \sigma_4, \dots, k_3, \sigma_2, \dots, n)$ :** Contributions from various pathes are given as

$$\begin{aligned} b_{II} \begin{bmatrix} \sigma_3 & \sigma_4 & \sigma_2 \\ k_1 & k_2 & k_3 \end{bmatrix} &= - \sum_{i=1}^{k_3} s_{\sigma_2 i} \\ b_{IV} \begin{bmatrix} \sigma_3 & \sigma_4 & \sigma_2 \\ k_1 & k_2 & k_3 \end{bmatrix} &= + \sum_{i=1}^{k_3} s_{\sigma_2 i} \end{aligned} \quad (2.18)$$

thus the total contribution is zero.

- **Type  $A(\widehat{1}, \widehat{5}, \dots, k_1, \sigma_2, \dots, k_2, \sigma_4, \dots, k_3, \sigma_3, \dots, n)$ :** Contributions from various pathes are given as

$$\begin{aligned} b_{III} \begin{bmatrix} \sigma_2 & \sigma_4 & \sigma_3 \\ k_1 & k_2 & k_3 \end{bmatrix} &= - \sum_{i=1, i \neq \sigma_3, \sigma_4}^{k_1} s_{\sigma_2 i} - \sum_{i=1}^{k_3} s_{\sigma_3 i} \\ b_{IV} \begin{bmatrix} \sigma_2 & \sigma_4 & \sigma_3 \\ k_1 & k_2 & k_3 \end{bmatrix} &= + \sum_{i=1, i \neq \sigma_3, \sigma_4}^{k_1} s_{\sigma_2 i} \end{aligned} \quad (2.19)$$

Putting together we have

$$b^{(\sigma_2, \sigma_3, \sigma_4)} \begin{bmatrix} \sigma_2 & \sigma_4 & \sigma_3 \\ k_1 & k_2 & k_3 \end{bmatrix} = - \sum_{i=1}^{k_3} s_{\sigma_3 i} \quad (2.20)$$

- **Type  $A(\widehat{1}, \widehat{5}, \dots, k_1, \sigma_4, \dots, k_2, \sigma_2, \dots, k_3, \sigma_3, \dots, n)$ :** Contributions from various pathes are given as

$$\begin{aligned} b_{III} \begin{bmatrix} \sigma_4 & \sigma_2 & \sigma_3 \\ k_1 & k_2 & k_3 \end{bmatrix} &= - \sum_{i=1, i \neq \sigma_3}^{k_2} s_{\sigma_2 i} - \sum_{i=1}^{k_3} s_{\sigma_3 i} \\ b_{IV} \begin{bmatrix} \sigma_4 & \sigma_2 & \sigma_3 \\ k_1 & k_2 & k_3 \end{bmatrix} &= + \sum_{i=1, i \neq \sigma_3}^{k_2} s_{\sigma_2 i} \end{aligned} \quad (2.21)$$

Putting together we have

$$b^{(\sigma_2, \sigma_3, \sigma_4)} \begin{bmatrix} \sigma_4 & \sigma_2 & \sigma_3 \\ k_1 & k_2 & k_3 \end{bmatrix} = - \sum_{i=1}^{k_3} s_{\sigma_3 i} \quad (2.22)$$

- **Type  $A(\widehat{1}, \widehat{5}, \dots, k_1, \sigma_4, \dots, k_2, \sigma_3, \dots, k_3, \sigma_2, \dots, n)$ :** Only the fourth path gives nonzero contribution, thus we have

$$b^{(\sigma_2, \sigma_3, \sigma_4)} \begin{bmatrix} \sigma_4 & \sigma_3 & \sigma_2 \\ k_1 & k_2 & k_3 \end{bmatrix} = + \sum_{i=1}^{k_3} s_{\sigma_2 i} \quad (2.23)$$

Having established (2.15), (2.20), (2.22) and (2.23), we now show the cyclic sum of all  $b$  coefficients is zero by listing out following table (2.24):

	$A(\widehat{1}, 2, 3, 4, \widehat{5})$	$A(\widehat{1}, 3, 4, 2, \widehat{5})$	$A(\widehat{1}, 4, 2, 3, \widehat{5})$	cyclic sum
$A(\widehat{1}, \widehat{5}, k_1, 2, k_2, 3, k_3, 4)$	$+\sum_{i=1}^{k_3} s_{4i}$	$-\sum_{i=1}^{k_3} s_{4i}$	0	0
$A(\widehat{1}, \widehat{5}, k_1, 2, k_2, 4, k_3, 3)$	$-\sum_{i=1}^{k_3} s_{3i}$	$+\sum_{i=1}^{k_3} s_{3i}$	0	0
$A(\widehat{1}, \widehat{5}, k_1, 3, k_2, 2, k_3, 4)$	0	$-\sum_{i=1}^{k_3} s_{4i}$	$+\sum_{i=1}^{k_3} s_{4i}$	0
$A(\widehat{1}, \widehat{5}, k_1, 3, k_2, 4, k_3, 2)$	0	$+\sum_{i=1}^{k_3} s_{2i}$	$-\sum_{i=1}^{k_3} s_{2i}$	0
$A(\widehat{1}, \widehat{5}, k_1, 4, k_2, 2, k_3, 3)$	$-\sum_{i=1}^{k_3} s_{3i}$	0	$+\sum_{i=1}^{k_3} s_{3i}$	0
$A(\widehat{1}, \widehat{5}, k_1, 4, k_2, 3, k_3, 2)$	$+\sum_{i=1}^{k_3} s_{2i}$	0	$-\sum_{i=1}^{k_3} s_{2i}$	0

(2.24)

## 2.2 A general proof

Having above example for  $j = 5$ , now we give the proof for general  $j$ . The idea of the proof is following. First we use the generalized BCJ relation to write  $A(\widehat{1}, \sigma(2, \dots, j-1), \widehat{j}, j+1, \dots, n)$  as the sum of the form  $-\frac{1}{s_{\widehat{1}, \dots, i-1}} a A(\widehat{1}, \dots, \widehat{i}, \dots)$  and the form  $-\frac{1}{s_{\widehat{1}, \dots, i-1}} b A(\widehat{1}, \widehat{i}, \dots)$  with  $z$ -independent coefficients  $a, b$ . Terms with form  $-\frac{1}{s_{\widehat{1}, \dots, i-1}} a A(\widehat{1}, \dots, \widehat{i}, \dots)$  have the right large- $z$  behavior and they are safe to be neglected. Terms with form  $-\frac{1}{s_{\widehat{1}, \dots, i-1}} b A(\widehat{1}, \widehat{i}, \dots)$  are dangerous, so we need to show that after the cyclic sum these contributions are zero.

To do so we need to find the expression for coefficient  $b$ . To see the pattern we rewrite results (2.15), (2.20), (2.22) and (2.23) as

$$\begin{aligned}
& A_{lead}(\widehat{1}, 2, 3, 4, \widehat{5}, \dots, n) \\
&= -\frac{1}{s_{\widehat{1}234}} \sum_{\sigma \in \tilde{P}(O\{2,3,4\} \cup \emptyset^T)} \left( \sum_{\rho \in P(O\{\sigma\} \cup \{6, \dots, n\})} S_4(\rho) A(\widehat{1}, \widehat{5}, \rho, n) \right) \\
&+ \frac{1}{s_{\widehat{1}234}} \sum_{\sigma \in \tilde{P}(O\{2,3\} \cup O^T\{4\})} \left( \sum_{\rho \in P(O\{\sigma\} \cup \{6, \dots, n\})} S_3(\rho) A(\widehat{1}, \widehat{5}, \rho, n) \right) \\
&- \frac{1}{s_{\widehat{1}234}} \sum_{\sigma \in \tilde{P}(O\{2\} \cup O^T\{3,4\})} \left( \sum_{\rho \in P(O\{\sigma\} \cup \{6, \dots, n\})} S_2(\rho) A(\widehat{1}, \widehat{5}, \rho, n) \right). \tag{2.25}
\end{aligned}$$

where  $O^T(\alpha)$  means the reversed ordering of set  $\alpha$  and  $S_j(\rho)$  means the sum of  $s_{ji}$  for each element  $i$  in the ordering of all external legs (in this case the ordering  $\widehat{1}, \widehat{5}, \rho, n$ ) at the left hand side of element  $j$  in the given ordering  $\rho$ . The constraint permutations  $\tilde{P}(\alpha \cup \beta)$  means all permutations satisfying following three conditions: (1) relative ordering of elements in the set  $\alpha$  is kept; (2) relative ordering of elements in the set  $\beta$  is kept; (3) the last elements is always the last element of the set  $\alpha$ .

Having above observation, we can write down the general pattern as

$$A_{lead}(\widehat{1}, 2, 3, 4, \dots, i-1, \widehat{i}, \dots, n) \tag{2.26}$$

$$= -\frac{1}{s_{\widehat{1}2,\dots,i-1}} \sum_{j=2}^{i-1} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,i-1\})} (-)^{i-1-j} \left( \sum_{\rho \in P(O\{\sigma\} \cup O\{i+1,\dots,n-1\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n) \right)$$

Assuming this pattern is right, now we show the cyclic sum is zero, i.e., coefficient of a given ordering of  $\rho$  is zero. To do so, first we need to find where the ordering  $\rho$  can appear in the pattern (2.26). Since  $\rho \in P(O\{\sigma\} \cup O\{i+1, \dots, n-1\})$ , we see that there is one and only one ordering of  $\sigma$  can give the  $\rho$ . For given ordering  $\sigma$ , since  $\sigma \in \tilde{P}(O\{\alpha\} \cup O^T\{\beta\})$ , the last element of set  $\alpha$  is determined by the last element of ordering  $\sigma$ . Similarly, the first element of ordering  $\sigma$  must be the first element  $\alpha_{first} = \sigma_{first}$  of set  $\alpha$  or the last element  $\beta_{last} = \sigma_{first}$  of the set  $\beta$ . Because the amplitude is given by  $A(\widehat{1}, \alpha \cup \beta, \widehat{i}, i+1, \dots, n)$ , we see that there are two and only two amplitudes in the cyclic sum can give contributions to ordering  $\rho$  and they are  $A(\widehat{1}, \{\sigma_{first}, \tilde{\alpha}, \sigma_{last}\} \cup \{\beta\}, \widehat{i}, \dots, n)$  and  $A(\widehat{1}, \{\tilde{\alpha}, \sigma_{last}\} \cup \{\beta, \sigma_{first}\}, \widehat{i}, \dots, n)$ . Now from the general pattern (2.26) it is easy to see that these two contributions are same with opposite signs and their sum is zero.

Now let us prove the pattern (2.26). Using the generalized BCJ relation (1.9) (like the one given in (2.3)) and the leading part (2.26) for amplitudes with fewer legs between 1 and  $i$ , we can express the leading part of the amplitude  $A(\widehat{1}, 2, 3, \dots, i-1, \widehat{i}, \dots, n)$  as

$$\begin{aligned} & A_{lead}(\widehat{1}, 2, 3, \dots, i-1, \widehat{i}, \dots, n) \\ &= -\frac{1}{s_{\widehat{1}2\dots i-1}} \left[ \sum_{\rho \in P(O\{2,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} (S_2(\rho) + \dots + S_{i-1}(\rho)) A(\widehat{1}, \widehat{i}, \rho, n) \right. \\ & \quad \left. + \sum_{k=2}^{i-2} \sum_{j=2}^k (-1)^{k+1-j} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,k\})} \sum_{\rho \in P(O\{\sigma\} \cup O\{k+1,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n) \right]. \end{aligned} \quad (2.27)$$

We define  $\xi_{\rho(l)}$  to be the position of the  $l$ -th external particles in the permutation  $\rho$ . It is worth to notice that in the last line of (2.27), although the relative ordering inside each set  $\sigma$ ,  $\{k+1, \dots, i-1\}$  and  $\{i+1, \dots, n-1\}$  is kept, these three sets can have arbitrary relative ordering and we will divide relative ordering among three sets in following discussions.

Let us start from the ordering  $\xi_{\rho(2)} < \xi_{\rho(3)} < \dots < \xi_{\rho(i-1)}$ , which can come from the second line and the third line with  $j = k$  in equation (2.27). The contribution from the second line is given by

$$- \sum_{\rho \in P(O\{2,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} \frac{1}{s_{\widehat{1}2\dots i-1}} (S_2(\rho) + \dots + S_{i-1}(\rho)) A(\widehat{1}, \widehat{i}, \rho, n), \quad (2.28)$$

while contributions from the third line with  $j = k$  are given by

$$\sum_{\rho \in P(O\{2,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} \frac{1}{s_{\widehat{1}2\dots i-1}} S_k(\rho) A(\widehat{1}, \widehat{i}, \rho, n), \quad 2 \leq k \leq i-2 \quad (2.29)$$

Summing over  $k$  and the second line contributions together we have

$$-\frac{1}{s_{\widehat{1}2\dots i-1}} \sum_{\rho \in P(O\{2,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} S_{i-1}(\rho) A(\widehat{1}, \widehat{i}, \rho, n). \quad (2.30)$$

This is noting but just the  $j = i - 1$  part of (2.26).

Now we consider all permutations with following condition:  $\xi_{\rho(i-1)} > \xi_{\rho(i-2)} > \dots > \xi_{\rho(l)}$ , but  $\xi_{\rho(l-1)} > \xi_{\rho(l)}$  for a given  $l$ . In other words,  $l$  is the first one breaks the natural descendent ordering from  $i - 1$  to 2. It is easy to see that the second line and the third line with  $k < l - 1$  in (2.27) can not give such permutations. When  $k > l$ , from  $\sigma \in \tilde{P}(O\{2, \dots, j\} \cup O^T\{j+1, \dots, k\})$  (especially the  $O^T\{j+1, \dots, k\}$  part) we see that there is no contribution either. There are only two contributions coming from  $k = l - 1$  and  $k = l$ .

When  $k = l - 1$ , all  $2 \leq j \leq l - 1$  will contribute to this ordering. For a given  $j$ , we get

$$-\frac{1}{s_{\widehat{1}2\dots i-1}} (-1)^{l-j} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,l-1\})} \sum_{\rho \in P(O\{\sigma\} \cup O\{l,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n).$$

When  $k = l$ , only  $2 \leq j \leq l - 1$  contribute to this ordering and  $j = l$  does no contribute. For a given  $j$ , we get

$$\begin{aligned} & -\frac{1}{s_{\widehat{1}2\dots i-1}} (-1)^{l+1-j} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,l-1,l\})} \sum_{\rho \in P(O\{\sigma\} \cup O\{l+1,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n) \\ \rightarrow & -\frac{1}{s_{\widehat{1}2\dots i-1}} (-1)^{l+1-j} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,l-1\})} \sum_{\rho \in P(O\{\sigma\} \cup O\{l,l+1,\dots,i-1\} \cup O\{i+1,\dots,n-1\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n). \end{aligned}$$

where because we have required that  $\xi_{\rho(l)} < \xi_{\rho(l-1)}$  and  $\xi_{\rho(l)} < \xi_{\rho(l+1)}$ , the first line is equivalent to the second line. Thus for any given  $j$  ( $2 \leq j \leq l - 1$ ), contributions from cases  $k = l - 1$  and  $k = l$  will cancel each other.

Above cancelation will work for  $2 < l < i - 1$ . The case  $l = 2$  has been discussed in (2.30). For the case  $l = i - 1$ , i.e.,  $\xi_{\rho(i-2)} > \xi_{\rho(i-1)}$ , it can only come from the third line of (2.27) with  $k = i - 2$ , thus we have

$$\begin{aligned} & -\frac{1}{s_{\widehat{1}2\dots i-1}} \sum_{j=2}^{i-2} (-1)^{i-1-j} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,i-2\})} \sum_{\rho \in P(\{\sigma\} \cup O\{O\{i-1\} \cup O\{i+1,\dots,n-1\}\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n) \\ \rightarrow & -\frac{1}{s_{\widehat{1}2\dots i-1}} \sum_{j=2}^{i-2} (-1)^{i-1-j} \sum_{\sigma \in \tilde{P}(O\{2,\dots,j\} \cup O^T\{j+1,\dots,i-2,i-1\})} \sum_{\rho \in P(\{\sigma\} \cup O\{i+1,\dots,n-1\})} S_j(\rho) A(\widehat{1}, \widehat{i}, \rho, n). \quad (2.31) \end{aligned}$$

where again the first line is equivalent to the second line because we have required that  $\xi_{\rho(i-2)} > \xi_{\rho(i-1)}$ .

After summing the contributions from (2.30) and (2.31), we get the general pattern (2.26). From above proof, we can see that the study of cyclic sum is much more difficult than the study of permutation sum discussed in [22].

### 3. The adjacent case of cyclic sum

Having proved the conjecture (1.3) for non-adjacent case, we move to the adjacent case, which can happen when and only when the shifted pair is  $(1, n)$ . In this case, the boundary behavior is

$$\sum_{\text{cyclic } \alpha} A_n(\widehat{1}, \{\alpha(2, \dots, n-1)\}, \widehat{n}) \rightarrow \xi_{1\mu}(z) \xi_{n\nu}(z) \frac{G^{\mu\nu}(z)}{z} \quad (3.1)$$

Comparing to the proof of cyclic sum of non-adjacent case, adjacent case is much more simpler and essentially only the KK-relations are needed. Again we will use one example to demonstrate our idea of proof and then give the general proof.

#### 3.1 Five point example

We consider the five point case with  $(1, 5)$ -deformation. In this case, we have three amplitudes  $A(\widehat{1}, 2, 3, 4, \widehat{5})$ ,  $A(\widehat{1}, 3, 4, 2, \widehat{5})$  and  $A(\widehat{1}, 4, 2, 3, \widehat{5})$ . For these amplitudes, the first step is to use KK relation to write an amplitudes in terms of amplitudes with 4, 5 adjacent, thus we have

$$\begin{aligned} A(\widehat{5}, \{\widehat{1}, 3\}, 4, \{2\}) &= -A(\widehat{5}, \widehat{1}, 3, 2, 4) - A(\widehat{5}, \widehat{1}, 2, 3, 4) - A(\widehat{5}, 2, \widehat{1}, 3, 4) \\ &= -A(\widehat{1}, 2, 3, 4, \widehat{5}) - A(\widehat{1}, 3, 2, 4, \widehat{5}) - A(2, \widehat{1}, 3, 4, \widehat{5}), \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} A(\widehat{5}, \{\widehat{1}\}, 4, \{2, 3\}) &= A(\widehat{5}, \widehat{1}, 3, 2, 4) + A(\widehat{5}, 3, \widehat{1}, 2, 4) + A(\widehat{5}, 3, 2, \widehat{1}, 4) \\ &= A(\widehat{1}, 3, 2, 4, \widehat{5}) + A(3, \widehat{1}, 2, 4, \widehat{5}) + A(3, 2, \widehat{1}, 4, \widehat{5}). \end{aligned} \quad (3.3)$$

Having the expansion (3.2) and (3.3), we can observe that all six terms at the right hand side can be divided into following two types. The first type is the form  $A(\widehat{1}, \dots, 4, \widehat{5})$  where  $\widehat{1}, \widehat{5}$  are adjacent and second type is the form  $A(\dots, \widehat{1}, \dots, 4, \widehat{5})$  where  $\widehat{1}, \widehat{5}$  are not adjacent. For the second type, since  $\widehat{1}, \widehat{5}$  are not adjacent, each amplitude has the boundary behavior  $\xi_{1\mu}(z) \xi_{5\nu}(z) \frac{G^{\mu\nu}(z)}{z}$ , so it is the boundary behavior we try to prove for (3.1).

For the first type, although the boundary behavior of each amplitude is worse than the wanted (3.1) since  $\widehat{1}, \widehat{5}$  are adjacent, their sum is, in fact, zero which can be easily seen from

$$A(\widehat{1}, 2, 3, 4, \widehat{5}) + [-A(\widehat{1}, 2, 3, 4, \widehat{5}) - A(\widehat{1}, 3, 2, 4, \widehat{5})] + [A(\widehat{1}, 3, 2, 4, \widehat{5})] = 0 \quad (3.4)$$

#### 3.2 A general proof

Having above example, now we give a general proof for the adjacent case. The cyclic sum will be given by

$$I = A(\widehat{1}, 2, \dots, n-1, \widehat{n}) + \sum_{i=2}^{n-2} A(\widehat{1}, i+1, \dots, n-1, 2, \dots, i, \widehat{n}) \quad (3.5)$$

where in the first term,  $n-1, n$  are nearby while in other terms, they are not. As in demonstrated example, the first step is to use KK-relation to expand other terms in the form with  $n-1, n$  nearby

$$A(\widehat{1}, i+1, \dots, n-1, 2, \dots, i, \widehat{n}) = (-)^{i-1} \sum_{\sigma \in P(O(1, i+1, i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))} A(\widehat{n}, \sigma, n-1) . \quad (3.6)$$

Among all terms in (3.6), some will have the form  $A(\widehat{n}, \dots, \widehat{1}, \dots, n-1)$  which will give the wanted large- $z$  behavior, while other terms, which we will call the leading part, will have the form  $A(\widehat{n}, \widehat{1}, \dots, n-1)$ . These leading terms can be written as

$$A_{lead}(\widehat{1}, i+1, \dots, n-1, 2, \dots, i, \widehat{n}) = (-)^{i-1} \sum_{\sigma \in P(O(i+1, i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))} A(\widehat{1}, \sigma, n-1, \widehat{n}) . \quad (3.7)$$

where when  $i = n-2$ , (3.7) is reduced to just  $(-)^{n-3} A(\widehat{1}, n-2, n-3, \dots, 2, n-1, \widehat{n})$ . Putting (3.7) back to (3.8) we have

$$\begin{aligned} I_{lead} &= A(\widehat{1}, 2, \dots, n-1, \widehat{n}) + \sum_{i=2}^{n-3} (-)^{i-1} \sum_{\sigma \in P(O(i+1, i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))} A(\widehat{1}, \sigma, n-1, \widehat{n}) \\ &\quad + (-)^{n-3} A(\widehat{1}, n-2, n-3, \dots, 2, n-1, \widehat{n}) \end{aligned} \quad (3.8)$$

To show  $I_{lead} = 0$ , it is important to notice that  $P(O(i+1, i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))$  means that either  $i+1$  or  $i$  at the second position, i.e.,

$$\begin{aligned} &\sum_{\sigma \in P(O(i+1, i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))} A(\widehat{1}, \sigma, n-1, \widehat{n}) \\ &= \sum_{\sigma \in P(O(i+1, i+2, \dots, n-2) \cup O(i-1, \dots, 2))} A(\widehat{1}, i, \sigma, n-1, \widehat{n}) \\ &\quad + \sum_{\sigma \in P(O(i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))} A(\widehat{1}, i+1, \sigma, n-1, \widehat{n}) . \end{aligned}$$

Using this observation we have

$$\begin{aligned} I_{lead} &= A(\widehat{1}, 2, \dots, n-1, \widehat{n}) \\ &\quad + (-) A(\widehat{1}, 2, 3, \dots, n-1, \widehat{n}) + \sum_{i=3}^{n-3} (-)^{i-1} \sum_{\sigma \in P(O(i+1, i+2, \dots, n-2) \cup O(i-1, \dots, 2))} A(\widehat{1}, i, \sigma, n-1, \widehat{n}) \\ &\quad + \sum_{i=2}^{n-4} (-)^{i-1} \sum_{\sigma \in P(O(i+2, \dots, n-2) \cup O(i, i-1, \dots, 2))} A(\widehat{1}, i+1, \sigma, n-1, \widehat{n}) + (-)^{n-4} A(\widehat{1}, n-2, \dots, 2, n-1, \widehat{n}) \\ &\quad + (-)^{n-3} A(\widehat{1}, n-2, n-3, \dots, 2, n-1, \widehat{n}) \end{aligned}$$

where we have split the  $i = 2$  term from the summation at the second line and  $i = n-3$  term from the summation at the third line. With above rewriting, it is easy to see the cancelation of the summation at the second line and third line, so  $I_{lead} = 0$ .

Having shown  $I_{lead} = 0$ , we know that

$$I = \sum_{\alpha, \beta, \text{ not empty}} c_i A(\widehat{1}, \alpha, \widehat{n}, \beta) . \quad (3.9)$$

Because for each term the  $1, n$  are not nearby, thus the large- $z$  behavior is

$$I \sim A(\widehat{1}, \alpha, \widehat{n}, \beta) \sim \xi_{1\mu}(z) \xi_{n\nu}(z) \frac{G^{\mu\nu}(z)}{z} \quad (3.10)$$

and we have finished the proof.

#### 4. The partial-ordered permutation sum

In this section, we will investigate the large  $z$ -behavior of following expression

$$I_{\text{partial}} \equiv \sum_{\sigma \in P(O\{2, \dots, l\} \cup \{l+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \dots, n) \quad (4.1)$$

where the sum is over all permutations of elements  $\{2, 3, \dots, i-1\}$  under one condition: the relative ordering in the subset  $\{2, \dots, l\}$  is kept. We will show that its large  $z$ -behavior is like

$$I_{\text{partial}} \sim \frac{1}{z} \sum_{\sigma' \in P(\{l+1, \dots, i-1\})} A(\widehat{1}, \sigma', \widehat{i}, \dots, n) \quad (4.2)$$

where  $1, i$  are not adjacent. For adjacent case, i.e., the partial-ordered permutation sum is given by

$$I_{\text{partial}}^{\text{adj}} \equiv \sum_{\sigma \in P(O\{2, \dots, l\} \cup \{l+1, \dots, n-1\})} A(\widehat{1}, \sigma, \widehat{n}), \quad (4.3)$$

where both sets  $\{2, \dots, l\}$  and  $\{2, \dots, l\}$  are not empty, using the  $U(1)$ -decoupling identity, one can write the above expression as

$$I_{\text{partial}}^{\text{adj}} = - \sum_{\sigma \in P(O\{2, \dots, l\} \cup \{l+1, \dots, n-2\})} A(\widehat{1}, \sigma, \widehat{n}, n-1) \quad (4.4)$$

Thus adjacent case is reduced to the nonadjacent case presented in (4.2). Relations (4.2) are new results of our paper and will be applied to the proof of (1.5).

Result (4.2) contains following special cases:

- When the set  $\{l+1, \dots, i-1\}$  is empty, we have the familiar result that the large  $z$ -behavior with non-adjacent deformation is one power of  $z$  better than the one with adjacent deformation

$$A(\widehat{1}, \dots, \widehat{i}, \dots, n) \sim \frac{1}{z} A(\widehat{1}, \widehat{i}, \dots, n). \quad (4.5)$$

- When the set  $\{2, \dots, l\}$  has only one element, the  $\sigma \in P(O\{2\} \cup \{3, \dots, i-1\})$  is nothing, but  $\sigma \in P(\{2, 3, \dots, i-1\})$ . Thus we reproduced the large  $z$ -behavior of permutation sum. Because this, we will assume that the first set has at least two elements.

These two special cases can be taken as the starting point of our inductive proof. Before giving the general proof, we will present two examples, where idea of our proof will be easier to understand.

#### 4.1 First example

We consider a simple example with  $O\{2, 3\}$  as the ordered set and  $\{4\}$  as the permuted set

$$T \equiv A(\widehat{1}, 2, 3, 4, \widehat{5}, \dots, n) + A(\widehat{1}, 2, 4, 3, \widehat{5}, \dots, n) + A(\widehat{1}, 4, 2, 3, \widehat{5}, \dots, n). \quad (4.6)$$

To see the  $z$ -dependence of  $T$ , we write down following two relations. The first one is the generalized BCJ relation (1.9) with set  $\beta = \{2, 3\}$

$$\begin{aligned} & s_{\widehat{123}} A(\widehat{1}, 2, 3, 4, \widehat{5}, \dots, n) + (s_{\widehat{123}} + s_{34}) A(\widehat{1}, 2, 4, 3, \widehat{5}, \dots, n) + (s_{\widehat{123}} + s_{34} + s_{24}) A(\widehat{1}, 4, 2, 3, \widehat{5}, \dots, n) \\ = & - \sum_{\sigma \in P(\{3\} \cup O\{6, \dots, n-1\})} (s_{\widehat{12}} + S_3(\sigma)) A(\widehat{1}, 2, 4, \widehat{5}, \sigma, n) \\ & - \sum_{\sigma \in P(\{3\} \cup O\{6, \dots, n-1\})} (s_{\widehat{12}} + s_{24} + S_3(\sigma)) A(\widehat{1}, 4, 2, \widehat{5}, \sigma, n) \\ & - \sum_{\sigma \in P(\{2, 3\} \cup O\{6, \dots, n-1\})} (S_2(\sigma) + S_3(\sigma)) A(\widehat{1}, 4, \widehat{5}, \sigma, n), \end{aligned} \quad (4.7)$$

while the second one is the fundamental BCJ relation for elements in the second set (here is just element 4)

$$\begin{aligned} & s_{4\widehat{1}} A(\widehat{1}, 4, 2, 3, \widehat{5}, \dots, n) + (s_{4\widehat{1}} + s_{42}) A(\widehat{1}, 2, 4, 3, \widehat{5}, \dots, n) + (s_{4\widehat{1}} + s_{42} + s_{43}) A(\widehat{1}, 2, 3, 4, \widehat{5}, \dots, n) \\ = & - \sum_{\sigma'' \in P(\{4\} \cup \{6, \dots, n-1\})} S_4(\sigma) A(\widehat{1}, 2, 3, \widehat{5}, \sigma, n). \end{aligned} \quad (4.8)$$

Here notation  $S_l(\sigma)$  has been defined under equation (2.25). Summing these two relations together, we get

$$\begin{aligned} & A(\widehat{1}, 2, 3, 4, \widehat{5}, \dots, n) + A(\widehat{1}, 2, 4, 3, \widehat{5}, \dots, n) + A(\widehat{1}, 4, 2, 3, \widehat{5}, \dots, n) \\ = & - \frac{1}{s_{\widehat{1234}}} \sum_{\sigma \in P(\{3\} \cup O\{6, \dots, n-1\})} (s_{\widehat{12}} + S_3(\sigma)) A(\widehat{1}, 2, 4, \widehat{5}, \sigma, n) \\ & - \frac{1}{s_{\widehat{1234}}} \sum_{\sigma \in P(\{3\} \cup O\{6, \dots, n-1\})} (s_{\widehat{12}} + s_{24} + S_3(\sigma)) A(\widehat{1}, 4, 2, \widehat{5}, \sigma, n) \\ & - \frac{1}{s_{\widehat{1234}}} \sum_{\sigma \in P(\{2, 3\} \cup O\{6, \dots, n-1\})} (S_2(\sigma) + S_3(\sigma)) A(\widehat{1}, 4, \widehat{5}, \sigma, n) \\ & - \frac{1}{s_{\widehat{1234}}} \sum_{\sigma \in P(O\{4\} \cup O\{6, \dots, n-1\})} S_4(\sigma) A(\widehat{1}, 2, 3, \widehat{5}, \sigma, n). \end{aligned} \quad (4.9)$$

The sum at the right-handed side can be divided into two types. The first type is

$$- \frac{s_{\widehat{12}}}{s_{\widehat{1234}}} \sum_{\sigma \in P(\{3\} \cup O\{6, \dots, n-1\})} [A(\widehat{1}, 2, 4, \widehat{5}, \sigma, n) + A(\widehat{1}, 4, 2, \widehat{5}, \sigma, n)] \sim \frac{1}{z} A(\widehat{1}, 4, \widehat{5}, \dots) \quad (4.10)$$



where we have used the result for  $\sigma \in P(O\{2\} \cup \{4\})$  of (4.1). The second type is remaining terms

$$-\frac{1}{s_{\widehat{1}234}}A(\widehat{1}, \dots, \widehat{5}, \dots) \sim \frac{1}{z}A(\widehat{1}, 4, \widehat{5}, \dots) . \quad (4.11)$$

Thus we have shown (4.2) for this example.

## 4.2 Second example

Previous example is a little bit simple. To see more clear the pattern of general proof, we consider another example

$$T \equiv \sum_{\sigma \in P(O\{2,3,4\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \dots, n). \quad (4.12)$$

Now we consider two types of relations. The first type is generalized BCJ relation with set  $\{\beta\} = \{2, 3, 4\}$  for each ordering of permutations  $\{5, 6\}$ . For example, with ordering 5, 6 we have

$$\begin{aligned} 0 = & \sum_{\sigma \in P(O\{2,3,4\} \cup O\{5,6\})} \left( \widehat{S}_2(\sigma) + \widehat{S}_3(\sigma) + \widehat{S}_4(\sigma) \right) A(\widehat{1}, \sigma, \widehat{7}, \dots, n) \\ & + \sum_{\gamma \in P(O\{4\} \cup O\{8, \dots, n-1\})} \sum_{\sigma \in P(O\{2,3\} \cup O\{5,6\})} \left( \widehat{S}_2(\sigma) + \widehat{S}_3(\sigma) + S_4(\gamma) \right) A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) \\ & + \sum_{\gamma \in P(O\{3,4\} \cup O\{8, \dots, n-1\})} \left( \widehat{S}_2(\sigma) + S_3(\gamma) + S_4(\gamma) \right) \sum_{\sigma \in P(O\{2\} \cup O\{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) \\ & + \sum_{\gamma \in P(O\{2,3,4\} \cup O\{8, \dots, n-1\})} (S_2(\gamma) + S_3(\gamma) + S_4(\gamma)) \sum_{\sigma \in P(O\{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) \end{aligned} \quad (4.13)$$

where to distinguish the  $z$ -dependence, we have used  $\widehat{S}_2(\sigma)$  to mean that it is  $s_{2\widehat{1}}$  in the sum while for  $S_2(\gamma)$ , it is  $s_{21} + s_{27}$  in the sum. Exchanging 5, 6 in (4.13) we get another relation.

The second type is the fundamental BCJ relation for each element 5, 6 and each possible relative ordering of  $\{2, 3, 4\}$  with remaining elements. For example, let us consider the fundamental BCJ relation for element 5. For a given ordering  $\sigma'$  in  $P(O\{2, 3, 4\} \cup O\{6\})$ , we have a fundamental BCJ relation

$$\sum_{\sigma \in P(O\{\sigma'\} \cup O\{5\})} \widehat{S}_5(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \dots, n) + \sum_{\gamma \in P(O\{5\} \cup O\{8, \dots, n-1\})} S_5(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n) = 0 \quad (4.14)$$

We should consider also fundamental BCJ relations for ordering  $\sigma'$  in  $P(O\{2, 3\} \cup O\{6\})$  and a given  $\gamma'$  in  $P(O\{4\} \cup O\{8, \dots, n-1\})$ , etc. Listing all them together we have

$$\begin{aligned} 0 = & \sum_{\sigma \in P(O\{\sigma'\} \cup O\{5\})} \widehat{S}_5(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \dots, n) + \sum_{\gamma \in P(O\{5\} \cup O\{8, \dots, n-1\})} S_5(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n), \quad \sigma' = P(O\{2, 3, 4\} \cup O\{6\}) \\ 0 = & \sum_{\sigma \in P(O\{\sigma'\} \cup O\{6\})} \widehat{S}_6(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \dots, n) + \sum_{\gamma \in P(O\{6\} \cup O\{8, \dots, n-1\})} S_6(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n), \quad \sigma' = P(O\{2, 3, 4\} \cup O\{5\}) \end{aligned}$$

$$\begin{aligned}
0 &= \sum_{\sigma \in P(O\{\sigma'\} \cup O\{5\})} \widehat{S}_5(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \gamma', n) + \sum_{\gamma \in P(O\{5\} \cup O\{\gamma'\})} S_5(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n), & \begin{cases} \sigma' = P(O\{2, 3, \} \cup O\{6\}) \\ \gamma' = P(O\{4\} \cup O\{8, \dots, n-1\}) \end{cases} \\
0 &= \sum_{\sigma \in P(O\{\sigma'\} \cup O\{6\})} \widehat{S}_6(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \gamma', n) + \sum_{\gamma \in P(O\{6\} \cup O\{\gamma'\})} S_6(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n), & \begin{cases} \sigma' = P(O\{2, 3, \} \cup O\{5\}) \\ \gamma' = P(O\{4\} \cup O\{8, \dots, n-1\}) \end{cases} \\
0 &= \sum_{\sigma \in P(O\{\sigma'\} \cup O\{5\})} \widehat{S}_5(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \gamma', n) + \sum_{\gamma \in P(O\{5\} \cup O\{\gamma'\})} S_5(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n), & \begin{cases} \sigma' = P(O\{2, \} \cup O\{6\}) \\ \gamma' = P(O\{3, 4\} \cup O\{8, \dots, n-1\}) \end{cases} \\
0 &= \sum_{\sigma \in P(O\{\sigma'\} \cup O\{6\})} \widehat{S}_6(\sigma) A(\widehat{1}, \sigma, \widehat{7}, \gamma', n) + \sum_{\gamma \in P(O\{6\} \cup O\{\gamma'\})} S_6(\gamma) A(\widehat{1}, \sigma', \widehat{7}, \gamma, n), & \begin{cases} \sigma' = P(O\{2\} \cup O\{5\}) \\ \gamma' = P(O\{3, 4\} \cup O\{8, \dots, n-1\}) \end{cases}
\end{aligned}$$

Summing all relations from these two types, we get

$$\begin{aligned}
T &= \sum_{\sigma \in P(O\{2,3,4\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \dots, n) \\
&= -\frac{1}{s_{\widehat{1}23456}} \left[ \sum_{\gamma \in P(O\{4\} \cup O\{8, \dots, n-1\})} (s_{\widehat{1}2356} + S_4(\gamma)) \sum_{\sigma \in P(O\{2,3\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) \right. \\
&\quad + \sum_{\gamma \in P(O\{3,4\} \cup O\{8, \dots, n-1\})} (s_{\widehat{1}256} + S_3(\gamma) + S_4(\gamma)) \sum_{\sigma \in P(O\{2\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) \\
&\quad + \sum_{\gamma \in P(O\{2,3,4\} \cup O\{8, \dots, n-1\})} (S_2(\gamma) + S_3(\gamma) + S_4(\gamma)) \sum_{\sigma \in P(\{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) \\
&\quad + \left\{ \sum_{\gamma \in P(O\{5\} \cup O\{8, \dots, n-1\})} S_5(\gamma) \sum_{\sigma \in P(O\{2,3,4\} \cup \{6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) + \{5 \leftrightarrow 6\} \right\} \\
&\quad + \left\{ \sum_{\gamma \in P(O\{5\} \cup O\{4\} \cup O\{8, \dots, n-1\})} S_5(\gamma) \sum_{\sigma \in P(O\{2,3\} \cup \{6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) + \{5 \leftrightarrow 6\} \right\} \\
&\quad \left. + \left\{ \sum_{\gamma \in P(O\{5\} \cup O\{3,4\} \cup O\{8, \dots, n-1\})} S_5(\gamma) \sum_{\sigma \in P(O\{2\} \cup \{6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n) + \{5 \leftrightarrow 6\} \right\} \right] \quad (4.15)
\end{aligned}$$

Now we can see the large  $z$ -behavior of  $T$ . For the first two lines at the right-handed side of (4.15), although  $S$ -factor is  $\frac{z}{s_{\widehat{1}23456}}$ , partial-ordered permutation sum of  $\sum_{\sigma \in P(O\{2,3\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n)$  and  $\sum_{\sigma \in P(O\{2\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \gamma, n)$  do give right behavior by induction. The third line is the pure permutation sum, but thanking the pre-factor  $\frac{1}{s_{\widehat{1}23456}}$ , we get the right result. For last three lines, although there is only one element 5 or 6 in the partial-ordered permutation sum, the pre-factor  $\frac{1}{s_{\widehat{1}23456}}$  provides the wanted reduction of power of  $z$ .

Overall we have

$$T \equiv \sum_{\sigma \in P(O\{2,3,4\} \cup \{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \dots, n) \sim \frac{1}{z} \sum_{\sigma \in P(\{5,6\})} A(\widehat{1}, \sigma, \widehat{7}, \dots, n). \quad (4.16)$$

### 4.3 General proof

Now let us consider the general case of (4.2). As did in previous example, we need to write down following two types of relations: the first type is just a single generalized BCJ relation (1.9) with  $\{\beta\} = \{2, \dots, l\}$ . For an arbitrary ordering of the elements in  $\{l+1, l+2, \dots, i-1\}$ , we get a BCJ relation of this type. The second type includes all fundamental BCJ relations for each element in the set  $\{l+1, l+2, \dots, i-1\}$ . For an arbitrary ordering of the element in  $O\{2, \dots, g\} \cup \{l+1, \dots, h-1, h+1, \dots, i-1\} \cup O\{i+1, \dots, n\}$  with  $2 \leq g \leq l^6$ , we get a BCJ relation of this type with  $\{\beta\} = \{h\}$ . Summing all the possible BCJ relations of the two types, we will arrive

$$\sum_{\sigma \in P(O\{2, \dots, l\} \cup \{l+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \dots, n) = -\frac{1}{s_{\widehat{1}, 2, \dots, i-1}} (I_1 + I_2 + I_3), \quad (4.17)$$

Where  $I_1, I_2, I_3$  are defined as

$$\begin{aligned} I_1 &= \sum_{g=2}^{l-1} \sum_{\gamma \in P(O\{g+1, \dots, l\} \cup O\{i+1, \dots, n-1\})} \left( s_{\widehat{1} 2 \dots g(l+1) \dots (i-1)} + \sum_{i=g+1}^l S_i(\gamma) \right) \sum_{\sigma \in P(O\{2, \dots, g\} \cup \{l+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n), \\ I_2 &= \sum_{\gamma \in P(O\{2, \dots, l\} \cup O\{i+1, \dots, n-1\})} (S_2(\gamma) + \dots + S_l(\gamma)) \sum_{\sigma \in P(\{l+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n), \\ I_3 &= \sum_{h=l+1}^{i-1} \sum_{g=2}^l \sum_{\gamma \in P(O\{h\} \cup O\{g+1, \dots, l\} \cup O\{i+1, \dots, n-1\})} S_h(\gamma) \sum_{\sigma \in P(O\{2, \dots, g\} \cup \{l+1, \dots, h-1, h+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n). \end{aligned}$$

Here  $I_1$  comes from both the first and the second types of BCJ relations.  $I_2$  can only come from the first type of BCJ relations, while  $I_3$  can only come from the second type of BCJ relation. All the  $I_1, I_2$  and  $I_3$  are written in terms of partial-ordered permutation sum, so we can use induction. For  $I_1$  part, although  $-\frac{s_{\widehat{1} 2 \dots g l+1 \dots i-1}}{s_{\widehat{1}, 2, \dots, i-1}} \sim z^0$ , the partial-ordered permutation sum gives

$$I_1 \sim \sum_{\sigma \in P(O\{2, \dots, g\} \cup \{l+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n) \sim \frac{1}{z} \sum_{\sigma \in P(\{l+1, \dots, h-1, h+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n). \quad (4.18)$$

For the  $I_2$  part, because the factor  $\frac{1}{s_{\widehat{1} \dots (i-1)}}$ , we have immediately

$$I_2 \sim \frac{1}{z} \sum_{\sigma \in P(\{l+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n). \quad (4.19)$$

---

<sup>6</sup>In the ordering, all  $\{l+1, \dots, h-1, h+1, \dots, i-1\}$  are at the left-handed side while  $\{i+1, \dots, n\}$  are at the right-handed side.

For the  $I_3$  part, although the number of elements in the partial-ordered permutation sum is reduced by one, the factor  $\frac{1}{s_{\widehat{1}\dots(i-1)}}$  provides the wanted another  $\frac{1}{z}$  reduction, thus we have

$$I_3 \sim \frac{1}{z^2} \sum_{\sigma \in P(\{l+1, \dots, h-1, h+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n) \sim \frac{1}{z} \sum_{\sigma \in P(\{l+1, \dots, h-1, h, h+1, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \gamma, n). \quad (4.20)$$

Summing all contributions from  $I_1$ ,  $I_2$  and  $I_3$ , we proved (4.2).

## 5. The combination of cyclic and permutation sums

In this section, we will prove

$$\sum_{\sigma \in Z(\{2, \dots, i-1\})} \sum_{\rho \in P(\{i+1, \dots, n\})} A(\widehat{1}, \sigma, \widehat{i}, \rho) \sim \frac{1}{z} \sum_{\rho \in P(\{i+1, \dots, n\})} A(\widehat{1}, \dots, \widehat{i}, \rho). \quad (5.1)$$

First let us consider following cyclic sum for a given ordering of  $\rho$

$$\begin{aligned} \sum_{\sigma \in Z(\{2, \dots, i-1\})} A(\widehat{1}, \sigma, \widehat{i}, \rho) &= A(\widehat{1}, 2, \dots, i-1, \widehat{i}, \rho) + \sum_{2 \leq k < i-1} A(\widehat{1}, k+1, \dots, i-1, 2, \dots, k, \widehat{i}, \rho) \\ &= A(\widehat{1}, 2, \dots, i-1, \widehat{i}, \rho) + \sum_{2 \leq k < i-1} \sum_{\sigma' \in O^T(2, \dots, k) \cup O\{k+1, \dots, i-2\}} (-1)^{k-1} A(\widehat{1}, \sigma', i-1, \widehat{i}, \rho) \\ &+ \sum_{2 \leq k < i-1} (-1)^{k-1} \sum_{2 \leq l < k} \sum_{\sigma' \in P(O\{k+1, \dots, i-2\} \cup O^T\{2, \dots, l\})} \sum_{\rho' \in P(O^T\{l+1, \dots, k\} \cup O\{\rho\})} A(\widehat{1}, \sigma', i-1, \widehat{i}, \rho'). \end{aligned} \quad (5.2)$$

where at the second and third lines we have used the KK-relation to expand  $A(\widehat{i}, \rho, \widehat{1}, k+1, \dots, i-2, i-1, 2, \dots, k)$  with  $(i-1)$  and  $i$  as pivots. In the expansion, there are two cases. The first case is that the set  $\rho$  and set  $\{2, \dots, i-1\}$  are separated by  $1, i$ , while all remaining belong to second case. The reason we make above separation is that all terms at the second line sum to zero by exactly same reason as we show the leading part contribution is zero in the section 3.2.

Having simplified the cyclic sum in (5.2), the sum combining cyclic and permutation can be written as

$$\begin{aligned} &\sum_{\sigma \in Z(\{2, \dots, i-1\})} \sum_{\rho \in P(\{i+1, \dots, n\})} A(\widehat{1}, \sigma, \widehat{i}, \rho) \\ &= \sum_{2 \leq k < i-1} (-1)^{k-1} \sum_{2 \leq l < k} \sum_{\sigma' \in P(O\{k+1, \dots, i-2\} \cup O^T\{2, \dots, l\})} \sum_{\rho' \in P(O^T\{l+1, \dots, k\} \cup \{i+1, \dots, n\})} A(\widehat{1}, \sigma', i-1, \widehat{i}, \rho') \end{aligned} \quad (5.3)$$

For given  $k$ ,  $l$  and  $\sigma'$ , the sum  $\sum_{\rho' \in P(O^T\{l+1, \dots, k\} \cup \{i+1, \dots, n\})} A(\widehat{1}, \sigma', i-1, \widehat{i}, \rho')$  is nothing but the partial-ordered permutation sum discussed in previous section. As we have shown in the previous section, the  $z$ -dependence should be suppressed by factor  $\frac{1}{z}$ . Thus the behavior (5.1) is proved.

## 6. Conclusion

In this note, using KK-relations and generalized BCJ relations, we have proved the large- $z$  behavior of cyclic sum and combination sum under BCFW recursion, which were first analyzed in [20, 21] through explicit Feynman diagram analysis. Our proof shows that these better convergent behaviors are natural consequences of the familiar boundary behavior of single pair BCFW-deformation, in the sense that they do not imply new nontrivial relations between amplitudes. However, as pointed by Boels and Isermann[20, 21], these behaviors do have important applications in understanding various properties of amplitudes at one-loop level.

A new result presented in our paper is the boundary behavior of partial-ordered permutation sum, which played a crucial role in our proof of the large- $z$  behavior of the combination sum. It would be interesting to see if the partial-ordered sum imposes further constraints on loop-level amplitudes similar to the vanishing conditions on box, triangle and bubble coefficients at one-loop imposed by cyclic and permutation sums.

We note that following the same reasoning in theories such as  $\mathcal{N} = 4$  SYM where KK and BCJ relations hold amplitudes are expected to present similar convergent behavior. However theories where gauge field couples to gravity require special attentions. In these cases despite KK relations remain valid it was argued[28] that BCJ relations cannot assume the usual form as in pure gauge theory. Having only KK relations at our disposal, we can derive the large- $z$  behavior of the cyclic sum in the adjacent case, while other types of sums in this theory await further studies.

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